2410Q Final Exam

Answer the questions in the space provided. If you need extra space, use the back or a spare sheet of paper and clearly indicate which problem is being continued on a separate page. No calculators, mobile devices, textbooks, or notes are allowed during the exam.

Name: ____

This exam has 9 questions for a total of 100 points. You have 2 hours to complete it.

Question	Points	Score
1	16	
2	15	
3	10	
4	10	
5	10	
6	9	
7	10	
8	10	
9	10	
Total:	100	

Laplace Transformations

- $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$ for $n = 0, 1, 2, \ldots$
- $\mathscr{L}\{e^{at}\} = \dfrac{1}{s-a}$
- $\mathscr{L}\{\sin(kt)\} = \dfrac{k}{s^2+k^2}$
- $\mathscr{L}\{\cos(kt)\} = \dfrac{s}{s^2+k^2}$
- $\mathscr{L}\{\sinh(kt)\} = \dfrac{k}{s^2-k^2}$
- $\mathscr{L}\{\cosh(kt)\} = \dfrac{s}{s^2-k^2}$
- $\mathscr{L}\{e^{at}f(t)\} = F(s-a)$
- $\mathscr{L}\{f(t-a)U(t-a)\} = e^{-as}F(s)$
- $\mathscr{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$
- $\mathscr{L}\{f * g\} = F(s)G(s)$ where $f * g = \displaystyle\int_0^t f(\tau)g(t-\tau)d\tau$.

1. Classify the following differential equations, and be sure to justify your answer. No credit will be given for answers without justification.

   (a) (4 points) $\frac{dy}{dx} = y(1 - y^2)$

   A. Separable B. Linear C. Exact D. Homogeneous E. Bernoulli

   (b) (4 points) $\vec{X}' = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \vec{X}$

   A. Homogeneous second order linear differential equation
   B. Nonhomogeneous second order linear differential equation
   C. Homogeneous system of first order linear differential equations
   D. Nonhomogeneous system of first order linear differential equations
   E. Autonomous Differential Equation

   (c) (4 points) $(3x^2 - 3y^2)dx + (x^2 + y^2)dy = 0$

   A. Separable B. Linear C. Exact D. Homogeneous E. Bernoulli

   (d) (4 points) $y'' - xy^2 - \tan(x) = 0$, $y(0) = 1$, $y(1) = 2$.

   A. Initial Value Problem
   B. Boundary Value Problem
   C. System of Differential Equations
   D. Autonomous Differential Equation
   E. Linear

2. Suppose there are two $10L$ buckets of salt water. Liquid is pumped from the first bucket to the second bucket at $2L/s$, and liquid is pumped from the second bucket to the first bucket in a separate pipe at $2L/s$.

   (a) (5 points) Write down a differential equation or system of differential equations modelling $A(t)$ =the amount of salt in the first bucket and $B(t)$ =the amount of salt in the second bucket.

   (b) (5 points) Solve for the general solution $A(t)$ and $B(t)$ as functions of t.

   (c) (5 points) Suppose that the first bucket starts with 50 grams of salt dissolved in the water, and the second bucket starts as pure water. What are the functions $A(t)$ and $B(t)$?

3. (10 points) Solve the following initial value problem

$$xy\frac{dy}{dx} = y^3 \qquad\qquad y(1) = 3$$

4. (10 points) Solve the following differential equation

$$y''' - y'' + y' - y = t^3$$

5. (10 points) Solve the following differential equation

$$\vec{X}' = \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix} \vec{X}$$

6. Consider the differential equation $\frac{dy}{dx} = y(y-1)e^y$

(a) (3 points) Sketch a phase portrait for this differential equation

(b) (3 points) Sketch a solution curve passing through the point $y(0) = \frac{1}{3}$.

(c) (3 points) If $y(t)$ is a solution to this differential equation passing through the point $y(0) = \frac{1}{3}$, what is $\lim_{t\to\infty} y(t)$?

7. (a) (5 points) What is the Laplace transformation of the piecewise function

$$f(t) = \begin{cases} 0 & 0 \le t < 1 \\ e^t & 1 \le t \end{cases}$$

(b) (5 points) Solve the initial value problem $y'' - 2y' + y = f(t)$, $y(0) = 1$, $y'(0) = 0$.

8. (10 points) Show that the vector functions $\vec{X}_1 = \begin{pmatrix} (t+1)e^t \\ -e^t \end{pmatrix}$ and $\vec{X}_2 = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$ are a fundamental set of solutions to $\vec{X}' = \begin{pmatrix} 2 & t \\ 0 & 1 \end{pmatrix} \vec{X}$.

9. Set up, but do not evaluate, a differential equation, system of differential equations, initial value problem, or boundary value problem modelling each of the following situations. In case you need it, the gravitational constant is $g = 9.8m/s^2$.

   (a) (5 points) Let $P(t)$ denote the population of Storrs, CT in the year t. The birth rate of Storrs is proportional to the total population, meanwhile, every year 100 residents of Storrs die and $\sqrt{P(t)}$ residents of Storrs move to Boston. In the year 2000, it was determined that 20,000 people lived in Storrs.

   (b) (5 points) Suppose you hang a 12kg ball from a spring. At rest, the spring is stretched 9.8 meters. The spring is then placed in a container filled with water, dampening the balls motion by a force twice the velocity of the ball. The ball is then lifted 2 meters above equilibrium and dropped (still underwater).

space for extra work